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Stability in quasi-uniform spaces and the inverse problem*

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Abstract

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We study the concept of a stable quasi-uniform space due to D. Doitchinov. In particular quasi-pseudo-metric spaces (X, d) whose associated quasi-uniformities $\mathcal{U}(d)$ or $\mathcal{U}(d^{-1})$ are stable are investigated.

Keywords: Stable quasi-uniformity, stable quasi-pseudo-metric, D-Cauchy filter, stable filter, hereditarily precompact, weakly submetacompact, subparacompact, quasi-developable, hereditarily separable, feebly compact.

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1. Introduction

In [5, 6] Doitchinov introduces the concept of a stable quasi-uniform space and develops an interesting theory of completeness for this class of quasi-uniform spaces. In this paper we continue his study of the concept of a stable quasi-uniform space. We observe that each quasi-uniform space $(X, \mathcal{U})$ whose conjugate $(X, \mathcal{U}^{-1})$ is hereditarily precompact is stable and that, on the other hand, the conjugate of each countably compact stable quasi-uniform space is hereditarily precompact. Initial quasi-uniformities induced by families of stable quasi-uniformities are shown to be stable. It follows e.g. that each topological space admits a finest (compatible) stable quasi-uniformity.

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In the second part of this paper we study those quasi-pseudo-metrizable spaces that admit a quasi-pseudo-metric d whose (quasi-pseudo-metric) quasi-uniformity $\mathcal{U}(d)$ is stable. These quasi-pseudo-metrics will be called *stable* in the following. We show that each Lindelöf stable quasi-pseudo-metric space is hereditarily separable and that each pseudo-$\aleph_1$-compact stable quasi-pseudo-metric space is separable. Completely regular pseudocompact stable quasi-pseudo-metric spaces are compact. Furthermore we prove that each regular σ-metacompact (weakly submetacompact) stable quasi-pseudo-metric space is paracompact (subparacompact). A regular stable quasi-pseudo-metric space is developable (pseudo-metrizable) provided that it is quasi-developable (has a σ-point-finite base). Various examples are presented to illustrate the obtained results. The questions whether each regular hereditarily separable stable quasi-pseudo-metric space is Lindelöf and whether there exists (in ZFC) a regular stable quasi-pseudo-metric space which is not countably metacompact remain open. For concepts not defined in this paper we refer the reader to [8, 14].

2. Stable quasi-uniformities

Let us recall the necessary definitions first. Following Fletcher and Hunsaker a filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is called *D-Cauchy* [4] provided that there exists a filter $\mathcal{G}$ on X such that for each $U \in \mathcal{U}$ there are $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $G \times F \subseteq U$. In this case we write $(\mathcal{G}, \mathcal{F}) \to 0$. A filter $\mathcal{F}$ on $(X, \mathcal{U})$ is called *stable* [3] provided that for each $U \in \mathcal{U}$ there exists an $A \in \mathcal{F}$ such that $A \subseteq U(B)$ whenever $B \in \mathcal{F}$. A quasi-uniform space is said to be *stable* [5, 6] if each of its D-Cauchy filters is stable. The quasi-uniformity of a stable quasi-uniform space will also be called *stable*. A quasi-pseudo-metric space (X, d) will be called *stable* provided that its quasi-pseudo-metric is stable. We recall that the quasi-pseudo-metric quasi-uniformity $\mathcal{U}(d)$ of a quasi-pseudo-metric space (X, d) is the filter generated on $X \times X$ by the base $\{U_\varepsilon : \varepsilon > 0\}$ where $U_\varepsilon = U_\varepsilon^d = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$ whenever ε is a positive real number. To motivate these definitions we discuss an example first.

Example 2.1. (a) Each uniform space is stable [6, Proposition 1].

(b) Let $\mathbb{R}$ be the set of real numbers and let $d(x, y) = y - x$ if $y \geqslant x$ and $d(x, y) = 1$ if $y < x$. Then the so-called quasi-metric Sorgenfrey line $(\mathbb{R}, d)$ is stable [5, 6]: Let $\mathcal{F}$ be a D-Cauchy filter on $(\mathbb{R}, \mathcal{U}(d))$. Then there exist a filter $\mathcal{G}$ on $\mathbb{R}$ and decreasing sequences $(G_n)_{n \in \mathbb{N}}$ and $(F_n)_{n \in \mathbb{N}}$ of subsets of $\mathbb{R}$ such that $G_n \in \mathcal{G}$, $F_n \in \mathcal{F}$ and $G_n \times F_n \subseteq U_{2^{-n}}$ whenever $n \in \mathbb{N}$. Note that G_1 and F_1 are bounded with respect to the usual metric $|\cdot|$ on $\mathbb{R}$ and that $g \leqslant f$ for each $g \in G_1$ and $f \in F_1$. Set $g_n = \sup G_n$ whenever $n \in \mathbb{N}$ and $g = \lim_{n \to \infty} g_n$ (with respect to $\mathcal{T}(d)$). Clearly $\mathcal{F}$ converges to g in $(\mathbb{R}, d)$. Moreover for each $B \in \mathcal{F}$ and $n \in \mathbb{N}$ the interval $(g, g + 2^{-n})$ is a subset of $U_{2^{-n}}(B)$. Hence using the notation of the definition above, for any $n \in \mathbb{N}$ we can set $A = [g, g + 2^{-n})$ if $g \in \bigcap_{B \in \mathcal{F}} U_{2^{-n}}(B)$, and $A = (g, g + 2^{-n})$ otherwise. We have shown that $(\mathbb{R}, d)$ is stable. Note that in this example the conjugate $(\mathbb{R}, d^{-1})$ of

$(\mathbb{R}, d)$ is stable, too. In the following we shall call a stable quasi-pseudo-metric (quasi-uniformity) whose conjugate is stable *doubly stable.*

Remark 2.2. A sequence $(x_n)_{n\in\mathbb{N}}$ of a quasi-pseudo-metric space (X, d) will be called a *D-Cauchy sequence* provided that the filter generated on X by $\{\{x_n : n\in\mathbb{N}, n\geqslant k\}: k\in\mathbb{N}\}$ is a D-Cauchy filter on $(X, \mathcal{U}(d))$. Stability of quasi-pseudo-metric spaces can be characterized by sequences. One easily checks that a quasi-pseudo-metric space (X, d) is stable if and only if each of its D-Cauchy sequences generates a stable filter. In fact the following criterion applies.

Lemma 2.3. *A quasi-pseudo-metric space* (X, d) *is stable if and only if for each D-Cauchy sequence* $(x_n)_{n\in\mathbb{N}}$ *on* X *and each* $m\in\mathbb{N}$ *the family* $(U_{2^{-m}}(x_n))_{n\in\mathbb{N}}$ *is point-finite at only finitely many points of* $\{x_n : n\in\mathbb{N}\}$.

Proof. Assume that each D-Cauchy sequence of a quasi-pseudo-metric space (X, d) satisfies the stated condition, but that there exists a D-Cauchy filter $\mathcal{F}$ on $(X, \mathcal{U}(d))$ which is not stable. Then there is an $m\in\mathbb{N}$ such that $\bigcap_{F\in\mathcal{F}} U_{2^{-m}}(F)\notin\mathcal{F}$. Furthermore there are a filter $\mathcal{G}$ on X and decreasing sequences $(G_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ of subsets of X such that $G_n\in\mathcal{G}$, $F_n\in\mathcal{F}$, $G_n\times F_n\subseteq U_{2^{-n}}$ and $F_n\setminus U_{2^{-m}}(F_{n+1})\neq\emptyset$ whenever $n\in\mathbb{N}$. For each $n\in\mathbb{N}$ choose $x_n\in F_n\setminus U_{2^{-m}}(F_{n+1})$. Then clearly the D-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ on X violates the stated condition—a contradiction. The converse is obvious. $\square$

Example 2.4. A quasi-uniformity for which each D-Cauchy filter with a countable base is stable need not be stable: Let $X=\omega_1$, let Δ be the diagonal of X and let $\mathcal{U}$ be the quasi-uniformity generated on $X\times X$ by $\{S_\alpha : \alpha<\omega_1\}$ where $S_\alpha=\Delta\cup(\{\beta\in\omega_1 : \beta$ is a limit ordinal, $\beta\geqslant\alpha\}\times\{\beta\in\omega_1 : \beta$ is a successor ordinal, $\beta\geqslant\alpha\})$ whenever $\alpha\in\omega_1$. Consider an arbitrary D-Cauchy filter $\mathcal{F}$ on X generated by a countable base $\{F_n : n\in\mathbb{N}\}$. Then there are a filter $\mathcal{G}$ on X, $F\in\mathcal{F}$, a cofinal subset B of ω_1 and for each $\alpha\in B$ a set $G_\alpha\in\mathcal{G}$ such that $G_\alpha\times F\subseteq S_\alpha$. Let $f\in F$. For any $\alpha\in B$ with $\alpha>f$ we have $S_\alpha^{-1}(f)=\{f\}=G_\alpha$ and $S_\alpha(G_\alpha)=\{f\}=F$. Hence $\mathcal{F}$ is stable, because it contains a singleton. On the other hand the filter generated on X by $\{\{\beta\in\omega_1 : \beta$ is a successor ordinal, $\beta\geqslant\alpha\} : \alpha\in\omega_1\}$ is a D-Cauchy filter which is not stable. Hence the quasi-uniformity $\mathcal{U}$ is not stable.

Further examples of stable quasi-uniformities can be constructed with the help of Propositions 2.5 and 2.12, which we are going to prove next. Recall that a quasi-uniform space $(X, \mathcal{U})$ is *precompact* if for each $U\in\mathcal{U}$ there is a finite subset F of X such that $U(F)=X$ and it is *totally bounded* if the uniformity $\mathcal{U}^*$ generated on X by $\mathcal{U}\cup\mathcal{U}^{-1}$ is precompact. The following result was independently obtained by J. Deák (private communication). It generalizes [3, Lemma 4.5].

Proposition 2.5. *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then each (ultra)filter on $(X, \mathcal{U})$ is stable if and only if $(X, \mathcal{U}^{-1})$ is hereditarily precompact.*

Proof. Let A be a subspace of $(X, \mathcal{U}^{-1})$ that is not precompact. Then there exists $U \in \mathcal{U}$ such that $A \setminus U^{-1}(E) \neq \emptyset$ for any finite subset E of A. Let $\mathcal{F}$ be a(n ultra)filter on X containing $\{A \setminus U^{-1}(E) \colon E \subseteq A,\ E \text{ finite}\}$. Then $\bigcap_{F \in \mathcal{F}} U(F) \subseteq X \setminus A$ and thus $\bigcap_{F \in \mathcal{F}} U(F) \notin \mathcal{F}$. Hence $\mathcal{F}$ is not stable. For the converse assume that $\mathcal{F}$ is a filter on $(X, \mathcal{U})$ that is not stable. Then there is $U \in \mathcal{U}$ such that $\bigcap_{F \in \mathcal{F}} U(F) \notin \mathcal{F}$. Similarly as in the proof of Lemma 2.3 we define a sequence $(x_n)_{n \in \mathbb{N}}$ of points of X: Choose inductively for each $n \in \mathbb{N}$, $F_n \in \mathcal{F}$ such that $F_n \setminus U(F_{n+1}) \neq \emptyset$ and $F_{n+1} \subseteq F_n$. Let $x_n \in F_n \setminus U(F_{n+1})$ whenever $n \in \mathbb{N}$. Consider $k, m \in \mathbb{N}$ with $k > m$. Then $x_k \in F_k \subseteq F_{m+1}$, but $x_m \in F_m \setminus U(F_{m+1})$. Thus $x_k \notin U^{-1}(x_m)$. Consequently the subspace $\{x_n : n \in \mathbb{N}\}$ of $(X, \mathcal{U}^{-1})$ is not precompact. □

Corollary 2.6. *A quasi-uniform space $(X, \mathcal{U})$ whose conjugate $(X, \mathcal{U}^{-1})$ is hereditarily precompact is stable.*

Remark 2.7. (a) The well-monotone covering quasi-uniformity (see [11, p. 20], compare [19] and [21, Corollary 8]) of a nonhereditarily compact space yields a simple example of a quasi-uniformity whose conjugate is hereditarily precompact, although it is not totally bounded.

(b) Since each second countable topological space admits a totally bounded quasi-pseudo-metric (compare [8, Proposition 7.2]), it is a consequence of Corollary 2.6 that each second countable space admits a doubly stable quasi-pseudo-metric. For later use we note that a second countable space need not be countably metacompact (see e.g. [8, Example 5.25]).

Recall that a quasi-uniform space $(X, \mathcal{U})$ is *Cauchy bounded* if each ultrafilter on X is D-Cauchy [13]. It is known that Cauchy bounded quasi-uniform spaces are precompact and that both compact and totally bounded quasi-uniform spaces are Cauchy bounded.

Corollary 2.8. *Let $(X, \mathcal{U})$ be a quasi-uniform space on which the elementary filter of any sequence is contained in a D-Cauchy filter (e.g. let $(X, \mathcal{U})$ be countably compact or Cauchy bounded). Then $\mathcal{U}$ is stable if and only if $\mathcal{U}^{-1}$ is hereditarily precompact.*

Proof. Assume that $\mathcal{U}^{-1}$ is not hereditarily precompact and that $\mathcal{U}$ satisfies the stated condition. Then there are $V \in \mathcal{U}$ and a sequence $(x_n)_{n \in \mathbb{N}}$ of points of X such that $x_n \notin V^{-1}(x_m)$ for each $n, m \in \mathbb{N}$ with $n > m$. Since no filter on X which contains $\{\{x_n : n \in \mathbb{N},\ n \geqslant k\} \colon k \in \mathbb{N}\}$ is stable, we conclude that $\mathcal{U}$ cannot be stable. The assertion is a consequence of Corollary 2.6. □

Remark 2.9. (a) Note that on a hereditarily precompact quasi-uniform space $(X, \mathcal{U})$ a stable ultrafilter $\mathcal{G}$ is Cauchy with respect to the supremum (quasi)-uniformity

$\mathcal{U}^*$ of $\mathcal{U}$ and $\mathcal{U}^{-1}$: For arbitrary $U \in \mathcal{U}$ there is $a \in \bigcap_{G \in \mathcal{G}} U(G)$ such that $U(a) \in \mathcal{G}$. Hence $U(a) \cap U^{-1}(a) \in \mathcal{G}$.

(b) On any quasi-uniform space $(X, \mathcal{U})$, an ultrafilter $\mathcal{G}$ is Cauchy with respect to $\mathcal{U}^*$ if (and only if) it is stable both in $(X, \mathcal{U})$ and $(X, \mathcal{U}^{-1})$: For arbitrary $U \in \mathcal{U}$, $x \in (\bigcap_{G \in \mathcal{G}} U(G)) \cap (\bigcap_{G \in \mathcal{G}} U^{-1}(G))$ implies that $U(x) \cap U^{-1}(x) \in \mathcal{G}$.

Corollary 2.10. (a) *Each Cauchy bounded hereditarily precompact stable quasi-uniform space is totally bounded.*

(b) *The Pervin quasi-uniformity is the unique stable quasi-uniformity on a hereditarily compact space.*

Proof. Recall that a quasi-uniform space $(X, \mathcal{U})$ is totally bounded if and only if both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are hereditarily precompact [18, Lemma 1.1]. Hence the first result is an immediate consequence of Corollary 2.8. Note that one could also use Remark 2.9(a) and [8, Proposition 3.14]. The second assertion follows from part (a) and the fact that the Pervin quasi-uniformity is the unique totally bounded quasi-uniformity on a hereditarily compact space (see [16]). □

We observe that [21, Example 7] is a hereditarily precompact stable quasi-metric space which is not totally bounded, although its conjugate is compact.

Let us call an entourage U of a quasi-uniform space $(X, \mathcal{U})$ *stable* provided that $\bigcap_{F \in \mathcal{F}} U(F) \in \mathcal{F}$ whenever $\mathcal{F}$ is a D-Cauchy ultrafilter on $(X, \mathcal{U})$.

Lemma 2.11. *A quasi-uniform space $(X, \mathcal{U})$ is stable if and only if it has a subbase consisting of stable entourages.*

Proof. Let $(X, \mathcal{U})$ be a quasi-uniform space having a subbase $\mathcal{S}$ consisting of stable entourages. Assume that there are a D-Cauchy filter $\mathcal{F}$ on $(X, \mathcal{U})$ and $U \in \mathcal{U}$ such that $\bigcap_{F \in \mathcal{F}} U(F) \notin \mathcal{F}$. There exist $n \in \mathbb{N}$ and $S_1, \ldots, S_n \in \mathcal{S}$ such that $\bigcap_{i=1}^n S_i \subseteq U$. Let $\mathcal{G}$ be an ultrafilter containing $\{F \setminus \bigcap_{F \in \mathcal{F}} U(F)\colon F \in \mathcal{F}\}$. Clearly $\mathcal{G}$ is a D-Cauchy filter on $(X, \mathcal{U})$ such that $\bigcap_{G \in \mathcal{G}} U(G) \notin \mathcal{G}$. However $\bigcap_{G \in \mathcal{G}} S_i(G) \in \mathcal{G}$ whenever $i \in \{1, \ldots, n\}$. Let $a \in \bigcap_{i=1}^n (\bigcap_{G \in \mathcal{G}} S_i(G))$. For each $i \in \{1, \ldots, n\}$, since $X \setminus S_i^{-1}(a) \notin \mathcal{G}$, $S_i^{-1}(a) \in \mathcal{G}$. Hence $U^{-1}(a) \in \mathcal{G}$ and thus $a \in \bigcap_{G \in \mathcal{G}} U(G)$. It follows that $\bigcap_{G \in \mathcal{G}} U(G) \in \mathcal{G}$—a contradiction. We conclude that $\mathcal{U}$ is stable. □

Proposition 2.12. *For any set X and any family $(X_i, \mathcal{U}_i)_{i \in I}$ of stable quasi-uniform spaces the coarsest quasi-uniformity $\mathcal{V}$ on X such that a given family of maps $f_i : X \to X_i$ $(i \in I)$ is quasi-uniformly continuous is stable.*

Proof. Let $\mathcal{F}$ be a D-Cauchy filter on $(X, \mathcal{V})$, let $i \in I$ and let $U \in \mathcal{U}_i$. Set $S = (f_i \times f_i)^{-1}(U)$. Since the filter $\mathcal{G}$ generated on X_i by $\{f_i(F)\colon F \in \mathcal{F}\}$ is a D-Cauchy filter on $(X_i, \mathcal{U}_i)$, we have $\bigcap_{F \in \mathcal{F}} U(f_i F) \in \mathcal{G}$. Thus $f_i^{-1}(\bigcap_{F \in \mathcal{F}} U(f_i F)) = \bigcap_{F \in \mathcal{F}} S(F) \in \mathcal{F}$. The assertion is a consequence of the preceding lemma. □

Corollary 2.13. (a) *Each subspace of a (doubly) stable quasi-uniform space is (doubly) stable.*

(b) *The product of any family of (doubly) stable quasi-uniform spaces is (doubly) stable.*

(c) *The supremum of any family of (doubly) stable quasi-uniformities on a given set is (doubly) stable.*

Remark 2.14. By Corollary 2.13(c), on each quasi-uniform space $(X, \mathcal{U})$ there exists a finest (doubly) stable quasi-uniformity $\mathcal{U}_s$ coarser than $\mathcal{U}$. Note that $\mathcal{U}_s$ is finer than the finest totally bounded quasi-uniformity contained in $\mathcal{U}$. Of course $\mathcal{U}_s$ is also finer than the finest uniformity coarser than $\mathcal{U}$. Moreover if $f:(X, \mathcal{U}) \to (Y, \mathcal{V})$ is a quasi-uniformly continuous map between quasi-uniform spaces, then $f:(X, \mathcal{U}_s) \to (Y, \mathcal{V}_s)$ is quasi-uniformly continuous.

Corollary 2.15. *Each topological space admits a finest (doubly) stable quasi-uniformity.*

We finish this section with two examples illustrating some further aspects of the concept of a stable quasi-uniformity.

Example 2.16. (a) Consider the subspace $Y = \{(x, -x)\colon\ x \in \mathbb{R}\} \cup \bigcup_{n=1}^{\infty} (k/n, (1-k)/n)\colon k$ is integer$\}$ of the (quasi-metric) Sorgenfrey plane (compare Example 2.1(b)). It is easy to check that Y is a (doubly stable quasi-metric) locally compact nonmetrizable Moore space.

(b) Note that each left K-sequentially complete (see e.g. [21]) quasi-metric space (X, d) whose conjugate is stable is strong: We have to show that $\mathcal{T}(d) \subseteq \mathcal{T}(d^{-1})$. However this is clear, since for each $x \in X$ and each sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $\lim_{n\to\infty} d(x_n, x) = 0$, we deduce by Corollary 2.8 (applied to the compact subspace $\{x\} \cup \{x_n : n \in \mathbb{N}\}$ of (X, d^{-1})) and by [21, Theorem 3(c)] that $(x_n)_{n \in \mathbb{N}}$ has a left K-Cauchy subsequence converging (necessarily) to x in (X, d).

Example 2.17. We give an example of a stable quasi-uniformity which is not the supremum of stable quasi-pseudo-metric quasi-uniformities: Let $X = \omega_1$ and let Δ be the diagonal of X. For each limit ordinal $l \in \omega_1$ choose a sequence $(\gamma_{ln})_{n \in \mathbb{N}}$ of distinct successor ordinals bigger than l. Set $S_\alpha = \Delta \cup \bigcup_{\omega_1 > l \geq \alpha} (\{l\} \times \{\gamma_{ln}\colon\ n \in \mathbb{N}\})$ whenever $\alpha \in \omega_1$ and let $\mathcal{U}$ be the quasi-uniformity generated by $\{S_\alpha : \alpha \in \omega_1\}$ on X. Consider an arbitrary D-Cauchy filter $\mathcal{F}$ on $(X, \mathcal{U})$. There exists a filter $\mathcal{G}$ on X such that $(\mathcal{G}, \mathcal{F}) \to 0$. Then there are $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $G \times F \subseteq S_0$. Let $g \in G$. Since $F \subseteq S_0(g)$, there is $\beta \in \omega_1$ such that $\sup F < \beta$. For $G' \in \mathcal{G}$ and $F' \in \mathcal{F}$ such that $F' \subseteq F$ and $G' \times F' \subseteq S_\beta$, and any $f' \in F'$ we have that $S_\beta^{-1}(f') = \{f'\} = G'$ and $S_\beta(G') = \{f'\} = F'$. Consequently $\mathcal{F}$ contains a singleton and is stable. We have shown that $(X, \mathcal{U})$ is stable.

Assume now that $\mathcal{U}$ is the supremum of stable quasi-pseudo-metric quasi-uniformities on X. Then there exists a stable quasi-pseudo-metric d on X such that $S_0 \in \mathcal{U}(d)$ by Corollary 2.13(c). Furthermore for each $m \in \mathbb{N}$ there exists $\alpha_m < \omega_1$

such that $S_{\alpha_m} \subseteq U^d_{2^{-m}}$. Let $l_0 < \omega_1$ be a limit ordinal greater than $\sup\{\alpha_m : m \in \mathbb{N}\}$. Since $S_{l_0} \subseteq \bigcap_{m=1}^{\infty} S_{\alpha_m}$, we see that $d(l_0, \gamma_{l_0 n}) = 0$ whenever $n \in \mathbb{N}$. However $\bigcap_{k=1}^{\infty} S_0(\{\gamma_{l_0 n} : n \in \mathbb{N},\ n \geq k\}) = \emptyset$ and thus d is not stable—a contradiction. We conclude that $\mathcal{U}$ is not the supremum of stable quasi-pseudo-metric quasi-uniformities.

3. Stable quasi-pseudo-metrics

To begin it may be interesting to note that a regular quasi-metric space (X, d) is metrizable provided that $\bigcap_{B \in \mathcal{F}} U^{-1}_{2^{-n}}(B) \in \mathcal{F}$ whenever $n \in \mathbb{N}$ and $\mathcal{F}$ is a convergent filter on (X, d). This observation follows from the proof of [21, Proposition 4], because in such a space for each convergent sequence $(y_n)_{n \in \mathbb{N}}$ the subspace $\{y_n : n \in \mathbb{N}\}$ is precompact.

Our main tools when studying stable quasi-pseudo-metrics are Lemmas 3.1, 3.7 and 3.11 proved below. During the presentation of their consequences it will become clear that many quasi-metric spaces discussed in the literature do not admit a stable quasi-metric; e.g. the Michael line (it has a σ-point-finite base), the example discussed by Aull in [1] (a paracompact Hausdorff space with a σ-locally countable base is metrizable, see e.g. [2]) and the quasi-metric space constructed in [25] (it is weakly submetacompact, but not subparacompact).

In the following let κ be an infinite cardinal number. Recall that a family $(C_i)_{i \in I}$ of subsets of a topological space X is called *point* $< \kappa$ (*locally* $< \kappa$) at $x \in X$ provided that $|\{i \in I : x \in C_i\}| < \kappa$ (provided that there is a neighborhood U of x such that $|\{i \in I : U \cap C_i \neq \emptyset\}| < \kappa$).

Lemma 3.1. *Let $(C_i)_{i \in I}$ be a family of subsets of a stable quasi-uniform space $(X, \mathcal{U})$, let d be a quasi-pseudo-metric on X such that $\mathcal{U}(d) \subseteq \mathcal{U}$ and suppose that κ is regular and greater or equal than the character of $\mathcal{T}(\mathcal{U})$. If there exists a positive real number ε such that the family $(U^d_\varepsilon(C_i))_{i \in I}$ is point $< \kappa$ on some subspace A of X, then $(U^d_\delta(C_i) \cap A)_{i \in I}$ is locally $< \kappa$ in $(X, \mathcal{U})$ whenever δ is a positive real number smaller than ε.*

Proof. Assume the contrary. Then there is $x \in X$ such that $(U^d_\delta(C_i) \cap A)_{i \in I}$ is not locally $< \kappa$ at x. Furthermore there exists a subcollection $\{V_\alpha : \alpha < \kappa\} \subseteq \mathcal{U}$ such that $\{V_\alpha(x) : \alpha < \kappa\}$ is a neighborhood base at x. Without loss of generality we can assume that for each $\alpha < \kappa$ the set $\{\beta < \kappa : V_\beta = V_\alpha\}$ is cofinal. For each $\alpha < \kappa$ choose inductively an $i_\alpha \in I$ such that $V_\alpha(x) \cap U^d_\delta(C_{i_\alpha}) \cap A \neq \emptyset$ and such that $i_\alpha \neq i_\beta$ whenever $\beta < \alpha$. Consider the filter $\mathcal{F}$ generated on X by $\{V_\gamma(x) \cap \bigcup_{\kappa > \beta > \alpha} (U^d_\delta(C_{i_\beta}) \cap A) : \gamma, \alpha < \kappa\}$. Since $\mathcal{F}$ is convergent and $\mathcal{U}$ is stable, $\mathcal{F}$ is stable. Thus there are $y \in A$ and a cofinal subset D of κ such that $y \in U^d_\varepsilon(C_{i_\beta})$ whenever $\beta \in D$. However the family $(U^d_\varepsilon(C_i))_{i \in I}$ is point $< \kappa$ at y—a contradiction. Hence the family $(U^d_\delta(C_i) \cap A)_{i \in I}$ is locally $< \kappa$ at x. $\square$

Example 3.2. The set of the isolated points B of a stable quasi-metric space (X, d) is an F_σ-set in (X, d): Let $n \in \mathbb{N}$ and $\mathcal{M}_n = \{\{x\}: U_{2^{-n}}(x) = \{x\}\}$. By Lemma 3.1, $\mathcal{M}_n$ is a locally finite (closed) collection in (X, d) and thus $\bigcup \mathcal{M}_n$ is closed. Hence $B = \bigcup_{n=1}^{\infty} (\bigcup \mathcal{M}_n)$ is an F_σ-set in (X, d).

Lemma 3.3. *Let κ be an infinite regular cardinal and let $\mathscr{C}$ be an open collection in a stable quasi-pseudo-metric space (X, d) that is point $<\kappa$ on a subspace A of X. Then there exists a collection $\bigcup_{n=1}^{\infty} \mathscr{R}_n$ of A-open sets refining $\{A \cap C: C \in \mathscr{C}\}$ such that each collection $\mathscr{R}_n$ is locally $<\kappa$ in (X, d) and such that for each $C \in \mathscr{C}$, $A \cap C = \bigcup_{n=1}^{\infty} R_n$ for appropriate sets $R_n \in \mathscr{R}_n$.*

Proof. Let $n \in \mathbb{N}$. For each $C \in \mathscr{C}$ set $H_n(C) = \{x \in C \cap A: U_{2^{-(n-1)}}(x) \cap A \subseteq C \cap A\}$. Then $U_{2^{-(n-1)}}(H_n(C)) \cap A \subseteq C \cap A$ whenever $C \in \mathscr{C}$. Therefore the family $(U_{2^{-(n-1)}}(H_n(C)) \cap A)_{C \in \mathscr{C}}$ is point $<\kappa$. By Lemma 3.1 we conclude that the collection $\mathscr{R}_n = \{U_{2^{-n}}(H_n(C)) \cap A: C \in \mathscr{C}\}$ is locally $<\kappa$ in (X, d). Note that $\bigcup_{n=1}^{\infty} (U_{2^{-n}}(H_n(C)) \cap A) = A \cap C$ whenever $C \in \mathscr{C}$. $\square$

Proposition 3.4. (a) *Each regular σ-metacompact stable quasi-pseudo-metric space X is paracompact.*

(b) *Each regular weakly submetacompact stable quasi-pseudo-metric space X is subparacompact.*

Proof. (a) Each open cover of X has a σ-locally finite open refinement by Lemma 3.3. Hence X is paracompact, since X is regular.

(b) Each open cover of X has a σ-locally finite refinement by Lemma 3.3. Hence X is subparacompact, since it is regular. $\square$

Proposition 3.5. (a) *Each regular stable quasi-pseudo-metric space X with a σ-point-finite base is pseudo-metrizable.*

(b) *Each regular quasi-developable stable quasi-pseudo-metric space X is developable.*

Proof. (a) By Lemma 3.3, X is a regular space with a σ-locally finite base.

(b) Since each quasi-developable space has a θ-base (e.g. [14, p. 479]), it follows from Lemma 3.3 that the (regular) space X has a σ-locally finite (closed) network. Hence X is perfect [14, p. 446] and thus developable (e.g. [14, p. 480]). $\square$

Analogously one proves the following results.

Corollary 3.6. (a) *Each stable quasi-pseudo-metric space (X, d) with a point-countable base has a σ-locally countable base.*

(b) *Each meta-Lindelöf stable quasi-pseudo-metric space is σ-para-Lindelöf.*

(c) *Each stable quasi-pseudo-metric space with a $\delta\theta$-base has a σ-locally countable network.*

(d) *Each open cover of a weakly submeta-Lindelöf stable quasi-pseudo-metric space has a σ-locally countable refinement.*

The proof of our next proposition relies on the following auxiliary result. The lemma describes another basic property of stable quasi-uniform spaces.

Lemma 3.7. *Let $(X, \mathcal{U})$ be a stable quasi-uniform space, let $x \in X$ and let $P, U \in \mathcal{U}$. Then there is a finite subset $F \subseteq P(x)$ such that $x \in \operatorname{int}_{\mathcal{T}(\mathcal{U})} U^{-1}(F)$.*

Proof. Assume the contrary. Since the filter $\mathcal{F}$ generated on X by $\{V(x) \setminus U^{-1}(a)$: $a \in P(x),\ V \in \mathcal{U}\}$ is stable, we have $\bigcap_{F \in \mathcal{F}} U(F) \in \mathcal{F}$, but $\bigcap_{F \in \mathcal{F}} U(F) \cap P(x) = \emptyset$—a contradiction. Hence there is a finite subset $F \subseteq P(x)$ such that $x \in \operatorname{int}_{\mathcal{T}(\mathcal{U})} U^{-1}(F)$. $\square$

Proposition 3.8. *The conjugate (X, d^{-1}) of a stable quasi-pseudo-metric space (X, d) is (hereditarily) weakly submetacompact. It is submetacompact if and only if it is countably metacompact.*

Proof. Let $\mathcal{C}$ be an open cover of (X, d^{-1}). There exists a refinement $\mathcal{R}$ of $\mathcal{C}$ such that $\mathcal{R} = \{U_{2^{-k}}^{-1}(x)$: $x \in A_k,\ k \in \mathbb{N}\}$ where $\{A_k : k \in \mathbb{N}\}$ is a cover of X. Let $n \in \mathbb{N}$. Set $G_n = \bigcup_{k=1}^{n} (\bigcup_{x \in A_k} U_{2^{-(k+1)}}^{-1}(x))$. We show that the cover $\mathcal{L}_n = \{G_n \cap U_{2^{-(n+1)}}^{-1}(x)$: $x \in G_n\}$ of the subspace G_n of (X, d^{-1}) has a θ-sequence $(\mathcal{G}_{nm})_{m \in \mathbb{N}}$ of $\mathcal{T}(d^{-1})$-open refinements. By a result of Worrell [12, Proposition 1.4] it suffices to prove that there exists a sequence $(\mathcal{H}_{nm})_{m \in \mathbb{N}}$ of $\mathcal{T}(d^{-1})$-open refinements of $\mathcal{L}_n$ such that for each $x \in G_n$ there is a sequence $(t(s))_{s \in \mathbb{N}}$ of natural numbers so that for each $s \in \mathbb{N}$, $\mathcal{H}_{nt(s+1)}$ is a pointwise W-refinement of $\mathcal{H}_{nt(s)}$ at x (i.e., there exists a finite subcollection $\mathcal{K}$ of $\mathcal{H}_{nt(s)}$ such that each member of $\{H$: $x \in H \in \mathcal{H}_{nt(s+1)}\}$ is contained in some member of $\mathcal{K}$).

To this end for each $m \in \mathbb{N}$ set $\mathcal{H}_{nm} = \{G_n \cap U_{2^{-(n+m)}}^{-1}(x)$: $x \in G_n\}$. Let $s \in \mathbb{N}$ and $x \in G_n$. By Lemma 3.7 there are $p \in \mathbb{N}$ such that $p \geq s+1$ and a finite subset F of G_n such that $G_n \cap U_{2^{-(p+n)}}(x) \subseteq U_{2^{-(s+1+n)}}^{-1}(F) \cap G_n$, since G_n is a stable subspace of (X, d). One easily checks that $\mathcal{H}_{np}$ is a pointwise W-refinement of $\mathcal{H}_{ns}$ at x. We conclude that Worrell's criterion is satisfied.

Clearly then the collection $\bigcup_{n,m=1}^{\infty} \mathcal{G}_{nm}$ is a weak θ-refinement of $\mathcal{C}$ in (X, d^{-1}). Therefore (X, d^{-1}) is weakly submetacompact. Assume now that (X, d^{-1}) is countably metacompact. There exists a $\mathcal{T}(d^{-1})$-closed cover $\{F_n : n \in \mathbb{N}\}$ of X such that $F_n \subseteq G_n$ whenever $n \in \mathbb{N}$. Setting $\mathcal{G}'_{nm} = \mathcal{G}_{nm} \cup \{C \setminus F_n : C \in \mathcal{C}\}$ whenever $n, m \in \mathbb{N}$ we obtain a θ-sequence of $\mathcal{T}(d^{-1})$-open refinements of $\mathcal{C}$. We conclude that (X, d^{-1}) is submetacompact if and only if (X, d^{-1}) is countably metacompact (see e.g. [12, Proposition 1.11]). $\square$

Example 3.16(a) below shows that a stable quasi-pseudo-metric space need not be weakly submetacompact. Note also that in light of Remark 2.7(b) we cannot omit the condition of countable metacompactness in the second part of Proposition 3.8.

Corollary 3.9. *Each regular doubly stable quasi-pseudo-metric space (X, d) is subparacompact and perfect. It has a G_δ-diagonal if it is a Hausdorff space.*

Proof. Subparacompactness of (X, d) is a consequence of Propositions 3.4(b) and 3.8. It remains to be shown that each open set G of (X, d) is an F_σ-set. However this is clear, since the cover $\{G'\colon G' \text{ is } \mathcal{T}(d)\text{-open and } \mathrm{cl}_{\mathcal{T}(d)} G' \subseteq G\}$ of G has a weak θ-refinement by Proposition 3.8 which according to Lemma 3.3 is refined by an in (X, d) σ-locally finite collection covering G.

Since by Corollary 2.13(b) $X \times X$ is perfect, the second assertion is obvious. □

Corollary 3.10 [22]. *Let S denote the Sorgenfrey line. Then the topological product S^ω is subparacompact and perfect.*

Proof. The assertion is a consequence of Example 2.1(b) and Corollaries 2.13(b) and 3.9. □

Lemma 3.11. *Each stable quasi-pseudo-metric space (X, d) contains a dense subset $\bigcup_{n=1}^\infty F_n$ such that the collection $\{\{x\}\colon x \in F_n\}$ is locally finite in (X, d) whenever $n \in \mathbb{N}$.*

Proof. For each $n \in \mathbb{N}$ define inductively and as long as possible a transfinite sequence $(x_\alpha)_{\alpha<\beta_n}$ such that for each $\beta < \beta_n$ we have $x_\beta \in X \setminus U_{2^{-n}}^{-1}(\{x_\alpha\colon \alpha < \beta\})$. Set $F_n = \{x_\alpha\colon \alpha < \beta_n\}$. Assume that $\{\{x_\alpha\}\colon \alpha < \beta_n\}$ is not locally finite at some point $x \in X$. Then there is a subsequence $(x_{\alpha_k})_{k\in\mathbb{N}}$ of $(x_\alpha)_{\alpha<\beta_n}$ converging to x such that $\alpha_k < \alpha_{k+1}$ whenever $k \in \mathbb{N}$—a contradiction to Lemma 2.3. Since $\bigcup_{x\in F_n} U_{2^{-n}}^{-1}(x) = X$ whenever $n \in \mathbb{N}$, it is clear that $\bigcup_{n\in\mathbb{N}} F_n$ is dense in (X, d). □

Corollary 3.12. *Each stable quasi-metric space contains a dense subspace which is the union of countably many closed discrete subspaces.*

Proposition 3.13. *A stable quasi-pseudo-metric space (X, d) in which each locally finite collection of open subsets is countable is separable. It is hereditarily separable if each locally finite collection of subsets is countable.*

Proof. (a) Assume that X is not hereditarily separable. Then X contains an uncountable left-separated subspace [14, p. 301]. Hence there exist $n \in \mathbb{N}$ and a transfinite sequence $(x_\alpha)_{\alpha<\omega_1}$ of points of X such that $x_\beta \notin U_{2^{-n}}^{-1}(x_\alpha)$ whenever $\alpha < \beta < \omega_1$. It follows from the proof of Lemma 3.11 that $\{\{x_\alpha\}\colon \alpha < \omega_1\}$ is locally finite in X. Hence X contains an uncountable locally finite collection of subsets.

(b) Assume now that each locally finite collection of open subsets of X is countable. First note that each pairwise disjoint open collection of X is countable:

Without loss of generality we consider an arbitrary collection $\mathcal{M} = \{U_{2^{-n(x)}}(x): x \in A\}$ where $A \subseteq X$ consisting of pairwise disjoint open balls. Then for each $n \in \mathbb{N}$ the collection $\mathcal{M}_n = \{U_{2^{-(n(x)+1)}}(x): x \in A \text{ and } n(x) = n\}$ is locally finite by Lemma 3.1. Hence $\mathcal{M}$ and $\bigcup_{n=1}^{\infty} \mathcal{M}_n$ are countable by our assumption.

Let $m \in \mathbb{N}$. By Lemma 3.7, for each $x \in X$ there are $s_{m(x)} \in \mathbb{N}$ and a finite subset $A_{m(x)}$ of $U_{2^{-m}}(x)$ such that $U_{2^{-s_{m(x)}}}(x) \subseteq \bigcup_{a \in A_{m(x)}} U^{-1}_{2^{-m}}(a)$. Since X satisfies the countable chain condition, there is a countable subset D_m of X such that $\bigcup_{d \in D_m} U_{2^{-s_{m(d)}}}(d)$ is dense in X. Set $S_m = \bigcup_{d \in D_m} A_{m(d)}$ and $S = \bigcup_{m=1}^{\infty} S_m$. Let us verify that the countable set S is dense in (X, d). Let $y \in X$ and $n \in \mathbb{N}$. There is $d \in D_{n+1}$ such that $U_{2^{-(n+1)}}(y) \cap U_{2^{-s_{(n+1)(d)}}}(d) \neq \emptyset$. Furthermore $U_{2^{-s_{(n+1)(d)}}}(d) \subseteq \bigcup_{a \in A_{(n+1)(d)}} U^{-1}_{2^{-(n+1)}}(a)$. Thus there is $a \in A_{(n+1)(d)}$ such that $a \in U_{2^{-n}}(y)$. Consequently $U_{2^{-n}}(y) \cap S \neq \emptyset$. We have shown that S is dense in X. $\square$

Recall that a topological space is called *pseudo-$\aleph_1$-compact* provided that each locally finite open collection is countable.

Corollary 3.14. (a) *For a stable quasi-pseudo-metric space the following conditions are equivalent: separable, ccc, weakly Lindelöf and pseudo-$\aleph_1$-compact.*

(b) *Each Lindelöf stable quasi-pseudo-metric space is hereditarily separable.*

Corollary 3.15. *A doubly stable quasi-pseudo-metric space is hereditarily separable if and only if it is (hereditarily) Lindelöf.*

Proof. It is known that a quasi-pseudo-metric space is hereditarily Lindelöf if and only if its conjugate is hereditarily separable [15, Theorem 4]. The result follows from Corollary 3.14(b). $\square$

Example 3.16. (a) Let (X, ρ) be a metric space and let $\{X_\alpha : \alpha < \beta\}$ be an increasing cover of X. Set $d(x, y) = 1$ if there exists $\alpha < \beta$ such that $x \in X_\alpha$ but $y \in X \setminus X_\alpha$, and set $d(x, y) = 0$ otherwise. Note that the quasi-pseudo-metric quasi-uniformity $\mathcal{U}(d)$ is stable, since $\mathcal{U}(d^{-1})$ is hereditarily precompact. Therefore the quasi-metric $s = \max\{\rho, d\}$ on X is also stable by Corollary 2.13(c). It follows that the perfectly collectionwise normal nonparacompact space constructed by Pol in [23] admits a stable quasi-metric. Note that it is not weakly submetacompact (e.g. [14, p. 377]) nor does it admit a doubly stable quasi-metric, since it is not subparacompact. It is well known that the construction discussed above can also be used to build a hereditarily separable stable quasi-metric Hausdorff space which is not Lindelöf (consider e.g. the space (X, s^{-1}) of [21, Example 5]).

(b) Let $(T, \leqslant)$ be a tree. Define a quasi-pseudo-metric ρ on T by setting $\rho(x, y) = 0$ if $x \geqslant y$ and $\rho(x, y) = 1$ otherwise. Using that for each $x \in T$, the set $\{y \in T: y \leqslant x\}$ is well ordered by $\leqslant$, one easily checks that (T, ρ) is a stable quasi-pseudo-metric space. We shall now exhibit a necessary and sufficient condition for the existence of a stable quasi-metric d on T such that d induces the usual *tree topology* of T.

As usual we say that a mapping $f: T \to \mathbb{R}$ is an *$\mathbb{R}$-embedding* of the tree $(T, \leqslant)$ if we have $f(x) < f(y)$ in $\mathbb{R}$ whenever $x < y$ in T and we say that $(T, \leqslant)$ is *$\mathbb{R}$-embeddable* if such a mapping f exists. Let us now show that a tree $(T, \leqslant)$ is $\mathbb{R}$-embeddable if and only if some stable quasi-metric induces the tree topology of $(T, \leqslant)$: If $f: T \to \mathbb{R}$ is an $\mathbb{R}$-embedding of $(T, \leqslant)$, then we can define a quasi-metric d on T by setting $d(x, y) = \max\{\rho(x, y), |f(x) - f(y)|\}$. By Corollary 2.13(c), d is a stable quasi-metric. It is easy to check that d induces the tree topology of $(T, \leqslant)$. If, conversely, there exists a stable quasi-metric on T which induces the tree topology, then Example 3.2 shows that the elements of T whose heights are successor ordinals form an F_σ-discrete (see [8, p. 132]) subset S_T of T. Since S_T is F_σ-discrete, it is the union of countably many antichains. According to [14, Lemma 9.7, p. 286] (see also [10]) T is $\mathbb{R}$-embeddable.

The $\mathbb{R}$-embeddable trees provide us with a consistent example of a nonperfect regular stable quasi-metric space: It is stated in [9, p. 61] by Hanazawa that under $\diamondsuit$ there exists an $\mathbb{R}$-embeddable Aronszajn tree which is not countably metacompact.

A topological space X is called *countably point-star preorthocompact* [20] provided that for each countable open cover $\mathscr{C}$ of X there exists a neighbornet V of X such that $V^2(x) \subseteq \mathrm{st}(x, \mathscr{C})$ whenever $x \in X$.

Proposition 3.17. *A countably point-star preorthocompact stable quasi-metric space X is countably metacompact and preorthocompact.*

Proof. Since by Corollary 3.12, X has a dense subspace which is the union of countably many closed discrete subspaces, the assertion is a consequence of the observation made in Remark (a) of [17, p. 19]. Furthermore X is preorthocompact, because each countably metacompact γ-space has this property [8, p. 168]. □

Problem 1. Is each regular hereditarily separable stable quasi-pseudo-metric space Lindelöf?

Problem 2. Is each (regular) Lindelöf stable quasi-pseudo-metric space hereditarily Lindelöf?

Problem 3. Is there in ZFC a regular stable quasi-metric space which is not countably metacompact?

Note that since every second countable space admits a doubly stable quasi-pseudo-metric, it is easy to give examples of Hausdorff stable quasi-metric spaces which are not countably metacompact (compare Remark 2.7(b)).

The following two major problems remain unresolved, too.

The Stability Problem. Characterize those quasi-pseudo-metrizable spaces that admit a stable quasi-pseudo-metric.

The Inverse Problem. Characterize those quasi-pseudo-metrizable spaces that admit a quasi-pseudo-metric whose conjugate is stable.

We finally show that a completely regular pseudocompact stable quasi-pseudo-metric space is compact. Our proof is based on two lemmas. First recall that a topological space X is called *feebly compact* [24] provided that each locally finite collection of open sets is finite.

Lemma 3.18. *The conjugate (X, d^{-1}) of each feebly compact stable quasi-pseudo-metric space (X, d) is precompact.*

Proof. By Corollary 3.14(a), (X, d) has a countable dense subset $D=\{x_n : n\in\mathbb{N}\}$ where it suffices to consider the case that the points x_n are distinct. (If D is finite, then (X, d^{-1}) is clearly precompact.) Assume that (X, d^{-1}) is not precompact. Then there is $m\in\mathbb{N}$ such that $U^{-1}_{2^{-m}}(F)\neq X$ whenever F is a finite subset of X. Inductively we define a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of nonempty open sets of (X, d). Set $y_1=x_1$. Let $p\in\mathbb{N}$ and assume that y_s $(s\leqslant p)$ and G_s $(s<p)$ are defined. Set $G_p=X\setminus\bigcup_{k=1}^{p} U^{-1}_{2^{-(m+1)}}(y_k)$ and let y_{p+1} be the point x_s with the smallest index $s\in\mathbb{N}$ such that $x_s\in X\setminus\bigcup_{k=1}^{p} U^{-1}_{2^{-(m+2)}}(y_k)$.

Since X is feebly compact, there is $x\in\bigcap_{p=1}^{\infty}\overline{G_p}$. Furthermore the convergent filter generated by $\{U_{2^{-n}}(x)\cap G_n : n\in\mathbb{N}\}$ on (X, d) is stable, because (X, d) is stable. Therefore there exists $r\in\mathbb{N}$ such that $U_{2^{-r}}(x)\cap G_r\subseteq\bigcap_{n=1}^{\infty} U_{2^{-(m+2)}}(G_n)$. Since $\emptyset\neq U_{2^{-r}}(x)\cap G_r\cap D$, there is $y\in(\bigcap_{n=1}^{\infty} U_{2^{-(m+2)}}(G_n))\cap D$. Moreover there exists $f\in\mathbb{N}$ such that $y\in U^{-1}_{2^{-(m+2)}}(y_f)$, because $D\subseteq\bigcup_{k=1}^{\infty} U^{-1}_{2^{-(m+2)}}(y_k)$. Since $y\in U_{2^{-(m+2)}}(G_f)$, we deduce that $y_f\in U_{2^{-(m+1)}}(G_f)$—contradicting the definition of G_f. We conclude that (X, d^{-1}) is precompact. $\square$

Before stating the second lemma used in the proof of our final proposition we want to recall the following two concepts. A topological space X is called *almost realcompact* (*closed complete*) provided that each maximal filter $\mathcal{G}$ of open (closed) sets on X with the property that $\bigcap\{\bar{G} : G\in\mathcal{G}\}$ has the countable intersection property, has a cluster point. It is known that each regular almost realcompact space is closed complete [7, Theorem 1.6]. The well-known space Ψ [14, p. 372, Example 4.4] is a separable quasi-metrizable Tychonoff space that is not almost realcompact.

Lemma 3.19. *Each regular separable quasi-pseudo-metric space (X, d) is closed complete. A regular quasi-pseudo-metric space (X, d) is almost realcompact if its conjugate (X, d^{-1}) is precompact.*

Proof. It seems convenient to prove these two results simultaneously. (The reader will note that our argument also applies in a Hausdorff γ-space instead of a regular quasi-pseudo-metric space.) Let $\mathcal{G}$ be a maximal filter of open (closed) sets on

(X, d) such that $\{\bar{G}\colon G \in \mathcal{G}\}$ has the countable intersection property. Fix $n \in \mathbb{N}$. There exists a finite (countable) subset A_{n+1} of X such that $X = U^{-1}_{2^{-(n+1)}}(A_{n+1})$. Since $\mathcal{G}$ has the finite (countable) intersection property, there is $f_{n+1} \in A_{n+1}$ such that $G \cap U^{-1}_{2^{-(n+1)}}(f_{n+1}) \neq \emptyset$ whenever $G \in \mathcal{G}$. By the maximality of $\mathcal{G}$ there exists $G_n \in \mathcal{G}$ such that $G_n \subseteq \overline{U^{-1}_{2^{-(n+1)}}(f_{n+1})}$. Let $x \in \bigcap_{n=1}^{\infty} \overline{G_n}$. Then $\overline{\{x\}} = \bigcap_{n=1}^{\infty} \overline{G_n} = \bigcap_{n=1}^{\infty} U^{-2}_{2^{-(n+1)}}(f_{n+1})$, because (X, d) is regular. Since $\{\bar{G}\colon G \in \mathcal{G}\}$ has the countable intersection property and X is regular, we conclude that $x \in \bigcap \{\bar{G}\colon G \in \mathcal{G}\}$. Hence (X, d) is almost realcompact (closed complete). □

Proposition 3.20. *Each regular feebly compact stable quasi-pseudo-metric space (X, d) is compact.*

Proof. By Lemma 3.18, the space (X, d^{-1}) is precompact. Hence each maximal filter of open sets on (X, d) has a cluster point by Lemma 3.19. Thus (X, d) is compact, since it is regular. □

Corollary 3.21. *Each completely regular pseudocompact stable quasi-pseudo-metric space is compact.*

References

[1] C.E. Aull, Topological spaces with a σ-point-finite base, Proc. Amer. Math. Soc. 29 (1971) 411–416.

[2] D.K. Burke, Refinements of locally countable collections, Topology Proc. 4 (1979) 19–27.

[3] Á. Császár, Extensions of quasi-uniformities, Acta Math. Acad. Sci. Hungar. 37 (1981) 121–145.

[4] D. Doitchinov, A concept of completeness of quasi-uniform spaces, Topology Appl. 38 (1991) 205–217.

[5] D. Doitchinov, Another class of completable quasi-uniform spaces, C.R. Acad. Bulgare Sci. 44 (1991) 5–6.

[6] D. Doitchinov, Stable quasi-uniform spaces and their completions, Preprint.

[7] N. Dykes, Generalizations of realcompact spaces, Pacific J. Math. 33 (1970) 571–581.

[8] P. Fletcher and W.F. Lindgren, Quasi-uniform Spaces, Lecture Notes in Pure and Applied Mathematics 77 (Marcel Dekker, New York, 1982).

[9] M. Hanazawa, Countable metacompactness and tree topologies, J. Math. Soc. Japan 35 (1983) 59–70.

[10] K.P. Hart, Characterizations of **R**-embeddable and developable ω_1-trees, Proc. Kon. Nederl. Akad. Wetensch. A 85 (1982) 277–283.

[11] H.J.K. Junnila, Covering properties and quasi-uniformities of topological spaces, Ph. D. thesis, Virginia Polytechnic Institute and State University, Blacksburg, VA (1978).

[12] H.J.K. Junnila, Three covering properties, in: G.M. Reed, ed., Surveys in General Topology (Academic Press, New York, 1980) 195–245.

[13] R.D. Kopperman, Total boundedness and compactness for filter pairs, Ann. Univ. Sci. Budapest 33 (1990) 25–30.

[14] K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984).

[15] H.P.A. Künzi, On strongly quasi-metrizable spaces, Arch. Math. 41 (1983) 57–63.

[16] H.P.A. Künzi, Topological spaces with a unique compatible quasi-proximity, Arch. Math. 43 (1984) 559–561.

[17] H.-P.A. Künzi, Kelley's conjecture and preorthocompactness, Topology Appl. 26 (1987) 13–23.

[18] H.-P.A. Künzi, Functorial admissible quasi-uniformities on topological spaces, Topology Appl. 43 (1992) 27–36.

[19] H.P.A. Künzi and N. Ferrario, Bicompleteness of the fine quasi-uniformity, Math. Proc. Cambridge Philos. Soc. 109 (1991) 167–186.
[20] H.P.A. Künzi and P. Fletcher, Some questions related to almost 2-fully normal spaces, Rocky Mountain J. Math. 15 (1985) 173–183.
[21] H.P.A. Künzi, M. Mršević, I.L. Reilly and M.K. Vamanamurthy, Convergence, precompactness and symmetry in quasi-uniform spaces, Math. Japon., to appear.
[22] D.J. Lutzer, Another property of the Sorgenfrey line, Compositio Math. 24 (1972) 359–363.
[23] R. Pol, A perfectly normal locally metrizable non-paracompact space, Fund. Math. 97 (1977) 37–42.
[24] C.T. Scarborough and A.H. Stone, Products of nearly compact spaces, Trans. Amer. Math. Soc. 124 (1966) 131–147.
[25] E.K. van Douwen and H.K. Wicke, A real, weird topology on the reals, Houston J. Math. 3 (1977) 141–152.